RIGIDITY OF COXETER GROUPS

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ABSTRACT. Let W be a Coxeter group acting properly discontinuously and cocompactly on manifolds N and M ($\partial M = \emptyset$) such that the fixed point sets of finite subgroups are contractible. Let $f : (N, \partial N) \to (M \times D^k, M \times S^{k-1})$ be a W-homotopy equivalence which restricts to a W-homeomorphism on the boundary. Under an assumption on the three dimensional fixed point sets, we show that then f is W-homotopic to a W-homeomorphism.

1. INTRODUCTION

A reflection on a connected manifold M is a locally linear involution on M whose fixed point set separates M into two components. A group W, generated by a set of reflections on a manifold, is called a reflection group if W acts effectively and properly discontinuously on M. The algebraic structure of reflection groups is very specific. The only relations in the group are induced by the intersections of the fixed submanifolds of the generators ([13]; [30]). Groups with such algebraic structure are called Coxeter groups because they were studied first in a paper by H. S. M. Coxeter ([12]).

More specifically, a Coxeter system (W, S) is a group W with a presentation

$$W = \langle s \in S : s^2 = (ss')^{m_{ss'}} = 1, s \neq s', m_{ss'} \in \{2, 3, \ldots, \infty\} \rangle.$$

In this paper we are interested in finitely generated (and therefore finitely presented) Coxeter groups.

In geometric topology, reflection groups appeared in the work of M. Davis ([13]; [14]), in the construction of closed aspherical topological manifolds not covered by Euclidean spaces. Using Davis' methods, G. Moussong constructed contractible, complete, non-positively curved (in the sense of Gromov), topological manifolds which are not homeomorphic to $\mathbb{R}^n$ ([25]).

Let W be a Coxeter group which admits a properly discontinuous, cocompact action on a manifold M (without boundary) such that the fixed point sets of finite subgroups are non-empty and contractible. We do not assume that the generators of W act by reflections. The main result (Theorem 5.3) of the paper is the topological rigidity of the W-action on M.

Theorem (Main Theorem). *Let N be a manifold with boundary on which W acts locally linearly, properly discontinuously and with the property that the fixed point sets of finite subgroups are non-empty and contractible. Let*

$$f : (N, \partial N) \rightarrow (M \times D^k, M \times S^{k-1})$$

be a W-homotopy equivalence which restricts to a W-homeomorphism on the boundary. If the three dimensional fixed point sets of N and $M \times D^k$ can be embedded in S^3, then f is W-homotopic to a W-homeomorphism.

We give an outline of the proof of the main theorem. The assumptions of the theorem imply that W is a virtual Poincaré duality group. By the results in [16], it splits as a product $W_1 \times W_2$, where W_1 is a a Coxeter group acting cocompactly by reflections on a contractible manifold and W_2 is a finite Coxeter group. The action of the group W_2 is trivial on all the spaces involved. Thus we can assume that W is a Coxeter group that acts cocompactly by reflections on a contractible manifold M'. Then the quotient space Q' (which can be identified with a fundamental domain of the action) is a manifold with boundary ([13]). The intersections $Q' \cap M'^s = \partial Q' \cap M'^s$ ($s \in S$), are manifolds with (possibly empty) boundary ([13]), called *panels* or *mirrors*. The panels of Q' and their non-empty intersections are contractible manifolds with (possibly empty) boundary. Comparing with M' enables us to show that the assumptions on $M \times D^k$ and N are sufficient to force W to act by reflections on both manifolds. Then any W-map $f : N \rightarrow M \times D^k$ is W-homotopic to a map g that restricts to a homotopy equivalence of the fundamental domains and preserves the panels. The map g induces homotopy equivalence on each non-empty intersection of the panels which is a compact contractible manifold. Using induction on the dimension of the manifolds, we can assume that the map is already a homeomorphism on the boundary. The generalized Poincaré conjecture implies that the map is homotopic to a homeomorphism, relative the boundary. Repeating the induction process, we construct a panel preserving homotopy from the restriction of g to a homeomorphism. Standard constructions outlined in [13] extend the homotopy already constructed to a W-homotopy from g (and therefore from f) to a W-homeomorphism. The assumption on the 3-dimensional fixed point sets is needed in the inductive process because the Poincaré conjecture is open in dimension 3. The result of our Main Theorem has been proved by E. Rosas ([26]) for right angled reflection groups (i.e. the exponents in the Coxeter relations are equal to either 2 or ∞).

The rigidity theorem for Coxeter groups proves the Borel-Quinn conjecture for Coxeter groups. The Borel-Quinn conjecture generalizes the rigidity phenomena expressed by the Borel conjecture to an equivariant setting. It has been formulated explicitly by Connolly and Koźniewski ([10]). Let Γ be a virtually torsion free, cocompact, discrete subgroup of the group of isometries of a complete, non-positively curved topological manifold $\tilde{M}$. Let Γ_0 be a normal torsion free subgroup of finite index, and $G = \Gamma / \Gamma_0$ the finite quotient. Let $\tilde{M}'$ be a contractible Γ-manifold. We assume that the actions of Γ on $\tilde{M}$ and $\tilde{M}'$ are properly discontinuous and cocompact, and that the fixed point set of each finite group of Γ is a contractible manifold, flatly embedded in a bigger fixed point set. The assumptions imply that there is a Γ-homotopy equivalence $F : \tilde{M}' \rightarrow \tilde{M}$. Let $M = \tilde{M}/\Gamma_0$, $M' = \tilde{M}'/\Gamma_0$ be the compact G-manifolds determined by the Γ actions on $\tilde{M}, \tilde{M}'$. Then the map F induces a G-homotopy equivalence $f : M' \rightarrow M$. We assume that the torsion of

f is zero in $Wh_G^{Top,\rho}(M)$ (the equivariant topological Whitehead group introduced in [28] and [29], for classifying isovariant G-h-cobordisms over M satisfying certain gap conditions).

Conjecture (Borel-Quinn Conjecture). *With the preceding notation the Γ-map F is Γ-homotopic to a Γ-homeomorphism.*

When the conjecture was stated in [10], there were three types of assumptions imposed: assumptions on the restriction of f to the fixed point sets of dimension less than or equal to 4, gap hypotheses, and assumptions on the finite subgroups of Γ of even order. The assumptions were necessary because of the examples given in [11], [36]. In the case of Coxeter groups, the gap hypotheses and the assumptions of subgroups of even order are not needed. We do need a condition on the three dimensional fixed point sets which is weaker than the hypotheses imposed in [10].

The condition on the torsion of f is a necessary condition. We will explain why the condition is not needed when $\Gamma = W$ is a Coxeter group acting on M by reflections. Let W_0 be a normal subgroup of finite index in W, $G = W/W_0$, and $M_0 = M/W_0$. In Section 5, we show that (Theorem 5.3)

$$Wh_G^{Top,\rho}(M_0 \times T^n) = 0,$$

where T^n is the n-dimensional torus with the trivial G-action. The proof is an adjustment of the ideas used in the proof of the main theorem.

In Section 2, we review the basic algebraic theory of Coxeter groups and the basic theory of panel spaces associated to Coxeter systems.

In Section 3, we show that the action of a Coxeter group on a universal space has the property that the fixed point sets are contractible if and only if all the panels are contractible. In this section, we extend the basic technical construction in [26] from the case of right angled reflection groups to general Coxeter groups.

In Section 4, we review the general theory of Coxeter groups when they act by reflections on polyhedra and manifolds, and the properties of the panel spaces determined by the action.

In Sections 5 and 6, we prove the topological rigidity theorem and the K-theory rigidity theorems for Coxeter groups, as outlined in the preceding paragraphs.

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2. Preliminaries on Coxeter Groups

We review the basic properties of Coxeter groups. Basic references are [3], [24], and [5] for a more geometric approach.

A *Coxeter system* (W, S) is a pair where W is a group and S is a set of generators of W, subject only to the relations

$$(ss')^{m(s,s')}, \ m(s,s) = 1, \ m(s,s') = m(s',s) \geq 2 \text{ if } s \neq s'.$$

Thus W is generated by a set of reflections, and the only relations in W are induced by the angles of intersection between the hyperplanes corresponding to the reflections. The group W is called a *Coxeter group* and the elements of S are called *simple reflections*. The conjugates of S are called *reflections* of the Coxeter system (W, S). We say that a Coxeter system (W, S) is the product of two Coxeter systems

$$(W, S) \cong (W_1, S_1) \times (W_2, S_2)$$

if $W \cong W_1 \times W_2$ and $S_i \subset S$, $i = 1, 2$. A Coxeter system is called *irreducible* if it cannot written as a non-trivial product of Coxeter systems. We will consider finitely generated (and therefore finitely presented) Coxeter groups. For $w \in W$, $\ell(w)$ denotes the length of w in the Coxeter presentation of W. The main properties of the length function in Coxeter groups are summarized in [24], Section 5.2.

Let $J \subseteq S$ and W_J be the subgroup of W generated by J. Then the pair (W_J, J) is again a Coxeter system ([3]; [24]). Subgroups of W of the form W_J ($J \subseteq S$) are called *parabolic subgroups*. Finite parabolic subgroups have the following universal property among the finite subgroups of W ([30]; [26], Lemma 2.1; [4], Proposition 1.3):

Proposition 2.1. *Let H be a finite subgroup of W. Then there is a finite parabolic subgroup W_J and an element $w \in W$ such that $H < wW_Jw^{-1}$.*

To a Coxeter system (W, S) we associate a partially ordered set

$$\mathcal{F}(W, S) = \{J \subseteq S : |W_J| < \infty\}$$

ordered by set inclusion. The set $\mathcal{F}(W, S)$ is isomorphic to the partially ordered set of finite parabolic subgroups of W.

The study of Coxeter group actions on spaces is connected with the theory of spaces equipped with a panel structure. We review the general theory of panel spaces. The basic definitions and properties of panel spaces are contained in [13] and [34].

Definition 2.1. An *S-panel structure* on a topological space Q is a locally finite family of closed subspaces $(Q_s)_{s \in S}$, indexed by a set S. The subsets Q_s are called *panels* of Q. The pair $(Q, (Q_s)_{s \in S})$ is called an *S-panel space*.

For each $q \in Q$, we write $S(q) = \{s \in S : q \in Q_s\}$. For each non-empty subset $J \subseteq S$, set

$$Q_J = \{q \in Q : J \subseteq S(q)\} = \bigcap_{s \in J} Q_s, \qquad Q_{\sigma(J)} = \bigcup_{s \in J} Q_s.$$

We define $Q_\emptyset = Q$. The formal boundary Q is defined to be the union of all the panels:

$$\delta Q = \bigcup_{s \in S} Q_s.$$

The subspaces Q_J are called *faces*. We will consider panel structures with finitely many panels.

We fix a Coxeter system (W, S). S-panel structures on spaces can be used for the construction of spaces equipped with a W-action. Let $(Q, (Q_s)_{s \in S})$ be an S-panel space. Define

$$\mathcal{U}(W, Q) = W \times Q/\sim$$

where $(w, q) \sim (w', q')$ if and only if $q = q'$ and $w^{-1}w' \in W_{S(q)}$. The space $\mathcal{U}(W, Q)$, with the quotient topology, is called the *universal space* of the pair (W, Q). We write $[w, q]$, $w \in W$, $q \in Q$, for elements in $\mathcal{U}(W, Q)$. There is a natural embedding

$$i : Q \to \mathcal{U}(W, Q), \quad q \to [e, q].$$

The group W acts on $\mathcal{U}(W, Q)$ by left multiplication in the first coordinate. The isotropy group of the point $[e, q]$ is $W_{S(q)}$, because only the generators in $S(q)$ fix $[e, q]$. Therefore the isotropy group of a general element $[w, q]$ is $wW_{S(q)}w^{-1}$. The space $\mathcal{U}(W, Q)$ has the following universal property ([34], p. 1088).

Proposition 2.2. *For any W-space M and any map $f : Q \to M$ such that $sf(q) = f(q)$, for all $s \in S$, $q \in Q_s$, there is a unique W-equivariant map $f^* : \mathcal{U}(W, Q) \to M$ such that $f^*([e, q]) = f(q)$.*

Definition 2.2. Let $(Q, (Q_s)_{s \in S}))$ and $(Q', (Q'_s)_{s \in S}))$ be S-panel spaces. An S-*panel map* is a continuous function $f : Q \to Q'$ that preserves panels, i.e., $f(Q_s) \subseteq Q'_s$, for all $s \in S$.

We will apply Proposition 2.2 to panel maps. Let $f : Q \to Q'$ be an S-panel map. Form the composition

$$f' : Q \xrightarrow{f} Q' \xrightarrow{i'} \mathcal{U}(W, Q'),$$

where i' is the natural inclusion. For $q \in Q_s$,

$$sf'(q) = s[e, f(q)] = [s, f(q)] = [e, f(q)] = f'(q).$$

Thus f' induces a W-map

$$\mathcal{U}(W, f) : \mathcal{U}(W, Q) \to \mathcal{U}(W, Q').$$

Summarizing, we have

Corollary 2.3. *Let $f : Q \to Q'$ be an S-panel map. Then there is a unique W-map $\mathcal{U}(W, f) : \mathcal{U}(W, Q) \to \mathcal{U}(W, Q')$ extending f.*

The conclusion of Corollary 2.3 can be used in the formulation of the naturality properties of the universal construction. Thus panel homeomorphisms induce W-homeomorphisms on the classifying spaces. Also, panel homotopies, i.e., S-panel maps $F : (Q \times I, (Q_s \times I)_{s \in S}) \to (Q', (Q'_s)_{s \in S})$, induce W-homotopies on the corresponding universal spaces.

Definition 2.3. Let $(Q, (Q_s)_{s \in S}))$ be an S-panel space. The panel structure on Q is called W-*finite* if $S(q) \in \mathcal{F}(W, S)$ for each $q \in Q$.

The W-action on a classifying space $\mathcal{U}(W, Q)$ is proper if and only if the S-panel structure of Q is W-finite ([13], Lemma 13.4; [34], p. 1089).

Let $(Q, (Q_s)_{s \in S})$ be an S-panel space with a W-finite structure, and let $\mathcal{U}(W, Q)$ denote the universal (W, Q)-space. We order the elements of W so that $w_1 = e$ and $w_i < w_j$ implies that $\ell(w_i) \leq \ell(w_j)$. Set

$$T_n(Q) = \bigcup_{i=1}^{n} w_i Q \subseteq \mathcal{U}(W, Q).$$

Lemma 2.4. *Let $n > 1$. Then*

$$T_{n-1}(Q) \cap w_n Q = w_n Q_{\sigma(B(w_n))},$$

where $B(w_n) = \{s \in S : \ell(w_n s) < \ell(w_n)\}$. Furthermore, $B(w_n) \in \mathcal{F}(W, S)$ for every n.

Proof. When $\mathcal{U}(W, Q)$ is a manifold, the proof is given in [13], Section 8. The same proof applies to the general case. □

For each $J \in \mathcal{F}(W, S)$, $w \in W$, we can have a natural choice for the coset representatives for wW_J.

Lemma 2.5. *Let $J \in \mathcal{F}(W, S)$. Then for any $w \in W$ there is a unique element $v \in wW_J$ of maximal length. Furthermore, w is the element of maximal length in wW_J if and only if $J \subseteq B(w)$.*

Proof. The element v is the product of the minimal coset representative for wW_J and the element of maximal length in W_J. The details appear in Lemma 1.7 in [16]. □

Definition 2.4. An S-panel space $(Q, (Q_s)_{s \in S})$ is called an *S-panel complex* if Q is a CW-complex and all the faces are subcomplexes of Q.

For a Coxeter system (W, S) there is a natural choice of a polyhedron equipped with a W-finite S-panel structure such that the faces are subpolyhedra. We write $K_0(W, S)$ for the abstract simplicial complex with vertex set S and simplices the non-empty elements of $\mathcal{F}(W, S)$. Let $K(W, S)$ denote the cone of $K_0(W, S)$, i.e., $K(W, S) = K_0(W, S) \cup \{\emptyset\}$. We define an S-panel structure on the geometric realization $|K(W, S)|$ of the simplicial complex. The panel corresponding to $s \in S$ is the geometric realization of the closed star of s in $K_0(W, S)$ ([13]).

We can now express the connection between certain homotopy properties of the panels of Q and the classifying space $\mathcal{U}(W, Q)$.

Theorem 2.6. *For an S-panel complex $(Q, (Q_s)_{s \in S})$ the following are equivalent:*
  1. $\mathcal{U}(W, Q)$ *is contractible.*
  2. Q *is contractible and Q_J is acyclic for each $J \in \mathcal{F}(W, S)$.*
  3. *The union of chambers $T_n(Q)$ is contractible for all $n \geq 1$.*

Proof. When $\mathcal{U}(W, Q)$ is a manifold, the result is a special case of Theorem 10.1 and Corollary 10.3 in [13]. But the same methods can be applied to the general case. □

Vinberg in [34], showed that every Coxeter group is isomorphic to a linear Coxeter group, i.e., a Coxeter group that is a subgroup of a general linear group and acts by reflections on a cone in a vector space. Using Selberg's lemma we conclude that Coxeter groups are virtually torsion free, i.e. they contain a torsion free subgroup of finite index. Actually a Coxeter group (W, S) has finite virtual cohomological dimension

$$\mathrm{vcd}(W) \leq \dim |K(W, S)|.$$

More precise statements can be found in [1], [18], and [22].

3. Universal Spaces of Coxeter Systems

Let (W, S) be a Coxeter system and $(Q, (Q_s)_{s \in S})$ a W-finite S-panel complex. The main purpose of this section is the proof that $\mathcal{U}(W, Q)$ is a W-space of type $\mathcal{E}W$ if and only if Q and each face of Q are contractible.

We start with a basic definition ([9], [27]).

Definition 3.1. Let W be any discrete group. A space of type $\mathcal{E}W$ is a W-CW-complex such that the isotropy groups of the action are finite, and the fixed point sets of finite subgroups are non-empty and contractible. The space $\mathcal{E}W$ is unique up to W-homotopy ([9]).

We formalize the S-panel structures that we will need.

Definition 3.2. An S-panel structure on a space Q is called *admissible* if the panel structure is W-finite and Q and each non-empty face are contractible. It is called an *admissible S-panel complex structure*, if in addition, Q is an S-panel complex.

Let $K(W)$ be the W-complex constructed in [25]. More explicitly, $K(W)$ is defined as the universal space of the S-panel space $C|K_0(W,S)'|$, the cone on the geometric realization of the first barycentric subdivision of $K_0(W,S)$. The panel corresponding to s is the geometric realization of the closed star of s in $K_0(W,S)'$. Thus $C|K_0(W,S)'|$ is an admissible complex, by construction, and $K(W) = \mathcal{U}(W,C|K_0(W,S)'|)$. Also, $K(W)$ admits a locally flat metric of non-positive curvature such that W acts by isometries and reflections, i.e. the fixed point set of each generator $s{\in}S$ separates $K(W)$ in two components.

The authors are grateful to M. Davis for bringing to their attention the following result, which is an immediate consequence of the properties of the non-positive curvature metric on $K(W)$.

Lemma 3.1. *The W-complex $K(W)$ is a complex of type $\mathcal{E}W$.*

Proof. The action of W on $K(W)$ is proper, so the isotropy groups of the action are finite. We have to show that fixed point sets of finite subgroups are contractible. This is a general statement about the fixed point sets of groups of isometries of complexes of non-positive curvature. Let H be a finite subgroup of W. Let x, y be two points fixed by H. Then there is a unique geodesic α of minimal length joining x and y. Since H acts by isometries, α is a geodesic in $K(W)^H$. In [15], Proposition 1.4, a contraction $c : K(W){\times}I \rightarrow K(W)$ is defined by contracting along the geodesics. Therefore c restricts to a contraction on $K(W)^H$. $\qquad\square$

Lemma 3.2. *Let $(Q,(Q_s)_{s\in S})$ be an admissible S-panel space such that $(Q_J, \delta Q_J)$ satisfies the homotopy extension property for each $J{\in}\mathcal{F}(W,S)$. Then $\mathcal{U}(W,Q)$ is W-homotopy equivalent to $K(W)$. In particular, if $(Q,(Q_s)_{s\in S})$ is an admissible S-complex, $\mathcal{U}(W,Q)$ is a complex of type $\mathcal{E}W$.*

Proof. By Proposition 2.5 in [26], there is an S-panel map $f : Q \rightarrow C|K_0(W,S)'|$. Since all the faces in both complexes are contractible, f is an S-panel homotopy equivalence. The functorial properties of the universal construction (Corollary 2.3) imply that:

$$\mathcal{U}(W,f) : \mathcal{U}(W,Q) \rightarrow K(W)$$

is a W-homotopy equivalence. $\qquad\square$

We will show that the converse of Lemma 3.2 is true. We start with a W-finite S-panel complex Q such that the universal space $U = \mathcal{U}(W,S)$ is a space of type $\mathcal{E}W$, and show that Q is an admissible complex.

Let $J{\in}\mathcal{F}(W,S)$. We will describe U^{W_J}, the fixed point set of W_J, as a union of faces. We fix a total ordering of W as in Section 2.

Definition 3.3. With the above notation:
  1. Let $w{\in}W$, $K{\in}\mathcal{F}(W,S)$. We say that the pair (w,K) *represents a face* in U^{W_J} if $J \subset wW_Kw^{-1}$.
  2. An element $v{\in}W$ is called *J-maximal in the pair* (w,K) if (w,K) represents a face of U^{W_J} and v is the unique maximal element in the coset wW_K (Lemma 2.5).

We write

$$V_J = \{(v,K) : \exists w{\in}W, v \text{ is the } J\text{-maximal element in } (w,K)\}.$$

Define a total ordering on the set V_J as follows: $(v,K) \prec (v',K')$ if and only if

1. $v < v'$ in the total ordering of W, and
2. if $v = v'$, $K \supsetneqq K'$.

Remark 3.1. We summarize the obvious properties of the ordering.
1. The minimal element in the ordering is the pair (w_J, J), where w_J is the unique element of maximal length in W_J.
2. Lemma 2.5 implies that v is J-maximal in (v, K) if and only if $K \subseteq B(v)$.

Our goal is to describe the structure of U^{W_J} written as the union of faces indexed by V_J.

Lemma 3.3. *The space U^{W_J} can be written as union of faces vQ_K such that v is J-maximal in the pair (w, K).*

Proof. There are two immediate consequences of the definitions.
1. If (w, K) is a pair representing a face in U^{W_J}, then $wQ_K \subseteq U^{W_J}$.
2. If v is J-maximal in (w, K), then $vQ_K = wQ_K$.

These observations imply that each face vQ_K, with v the J-maximal element in (w, K) is contained in U^{W_J}.

For the other inclusion, let $x \in U^{W_J}$. Then the isotropy group of x is a conjugate of a parabolic subgroup, i.e., $W_x = wW_K w^{-1}$ for some $K \in \mathcal{F}(W, S)$, $w \in W$. Since x is fixed by W_J, it follows that $J \subset wW_K w^{-1}$ and the pair (w, K) represents a face in U^{W_J}. Thus $x \in wQ_K = vQ_K$, where v is the J-maximal element in (w, K). $\square$

Imitating the construction in Section 2, we define

$$T^J_{(v,K)} = \bigcup\nolimits_{(v',K') \preceq (v,K)} v'Q_{K'}.$$

Definition 3.4. An element (v, K) of V_J is called *reducible* if $vQ_K \subseteq T^J_{(v',K')}$ for some $(v', K') \npreceq (v, K)$. Otherwise (v, K) is called *irreducible*.

Remark 3.2. The pair $(v, B(v))$ is never irreducible, because, for any $K \subsetneqq B(v)$, $(v, K) \npreceq (v, B(v))$ and $vQ_{B(v)} \subsetneqq vQ_K$.

The following result is the analogue of Lemma 2.4.

Lemma 3.4. *Let $(v, K) \in V_J - \{(w_J, J)\}$, and let (v', K') denote the maximal element in V_J smaller than (v, K). If (v, K) is irreducible, then*

$$vQ_K \cap T^J_{(v',K')} = \bigcup\nolimits_{s \in B(v)-K} vQ_{K \cup \{s\}}.$$

Proof. First notice that, for all $s \in B(v)$, $K \cup \{s\} \in \mathcal{F}(W, S)$ by part (2) of Remark 3.1. Thus $Q_{K \cup \{s\}} \neq \emptyset$.

We start with the inclusion

$$\bigcup\nolimits_{s \in B(v)-K} vQ_{K \cup \{s\}} \subseteq vQ_K \cap T^J_{(v',K')}$$

For each $s \in B(v) - K$, $vQ_{K \cup \{s\}} \subseteq vQ_K$. Since $K \cup \{s\} \subseteq B(v)$, Lemma 2.5 implies that v is J-maximal in $(v, K \cup \{s\})$. Then $(v, K \cup \{s\}) \npreceq (v, K)$ and $vQ_{K \cup \{s\}} \subseteq T^J_{(K',v')}$.

For the other inclusion, let $vx \in vQ_K \cap T^J_{(K',v')}$. Then there is $(v'', K'') \npreceq (v, K)$ such that $vx \in vQ_K \cap v''Q_{K''}$. Then the intersection $Q_K \cap v^{-1}v''Q_{K''}$ is a non-empty face. If $Q_K \cap v^{-1}v''Q_{K''} = Q_K$, then $vQ_K \subseteq v''Q_{K''} \subset T^J_{(K',v')}$, contradicting the assumption that (v, K) is irreducible. Thus x must belong to a proper subface of Q_K. We distinguish two cases:

1. $v'' < v$ in the ordering of W. Let $v = w_n$ in the ordering of W. By assumption, $x \in vQ_K \cap T_{n-1}(Q)$. By Lemma 8.2 in [13], $x \in \bigcup_{s \in B(v)} Q_s$. Since x belongs to a proper subface of Q_K, there must be $s \in B(v) - K$ such that $x \in Q_{K \cup \{s\}}$.

2. $v = v''$, $K \subsetneqq K''$. Since v is a J-maximal element in (v, K''), $K'' \subseteq B(v)$. If $s \in K'' - K$ is a simple reflection, then

$$vx \in vQ_K \cap vQ_{K''} = vQ_{K''} \subseteq vQ_{K \cup \{s\}}.$$

$\square$

Lemma 3.5. *Let $(Q, (Q_s)_{s \in S})$ be a W-finite S-panel complex such that $\mathcal{U}(W, Q)$ is a complex of type $\mathcal{E}W$. Then the S-panel structure of Q is admissible.*

Proof. Let $U = \mathcal{U}(W, Q)$ as before. By assumption U is contractible, and by Theorem 10.1 in [13], Q is contractible and each face Q_J, $J \in \mathcal{F}(W, S)$, is acyclic. We will show that Q_J is contractible by reverse induction on the number of elements of J. If J is a maximal subset in $\mathcal{F}(W, S)$, then $Q_J = U^{W_J}$, which is contractible by assumption. If J is not maximal, then we assume that Q_K is contractible for each $K \in \mathcal{F}(W, S)$, $|J| < |K|$, and we will show that Q_J is contractible. By Lemma 3.4, U^{W_J} is given as an ascending union

$$U^{W_J} = \bigcup_{(v, K) \in V_J} T^J_{(v, K)}.$$

Fix an element $(v, K) \in V_J$. Let (v', K') be the maximal element smaller than (v, K). If (v, K) is reducible, then $T^J_{(v, K)} = T^J_{(v', K')}$. If not, then $K \subsetneqq B(v)$ (Remark 3.2), and $vQ_K \cap T^J_{(v', K')}$ is the union of faces of the form $vQ_{K \cup \{s\}}$, $s \in B(v) - K$. By assumption, all the faces $vQ_{K \cup \{s\}}$ are contractible. Their intersection

$$\bigcap_{s \in B(v) - K} vQ_{K \cup \{s\}} = vQ_{B(v)}$$

is also contractible by the induction hypothesis ($|J| \leq |K| \lneqq |B(v)|$). By Van Kampen's theorem, the map induced by inclusion, $\pi_1(T^J_{(v', K')}) \to \pi_1(T^J_{(v, K)})$, is an injection. Thus the inclusion induced map $\pi_1(Q_J) \to \pi_1(U^{W_J})$ is an injection ($Q_J = w_J Q_J$ corresponds to the minimal element in V_J). Therefore $\pi_1(Q_J) = 0$ and Q_J is contractible. $\square$

Summarizing, we get

Proposition 3.6. *The classifying space $\mathcal{U}(W, Q)$ of a finite S-panel complex Q is of type $\mathcal{E}W$ if and only if Q is admissible.*

4. Reflection Groups

We study properties of groups that act by reflections on manifolds. It turns out that all such groups are Coxeter groups ([13]; [30]). In that context, algebraic properties of the group can help in determining geometric properties of the action. By 'manifold' we mean 'topological manifold'.

Definition 4.1. Let M be a connected manifold, possibly with boundary.

1. A *reflection* on M is a locally smooth involution $r : M \to M$ with fixed point set M^r a submanifold of M that separates M into two components.

2. A discrete group W that acts on M effectively, properly discontinuously, and locally smoothly, and is generated by a set of reflections on M, is called a *reflection group* on M.

Let W be a reflection group on a connected manifold M^n. Then the action determines a set S of Coxeter generators of W, and the elements of W that act by reflections on M are exactly the reflections in the Coxeter system (W, S) ([13]). Furthermore, the action determines an S-panel space $(Q, (Q_s)_{s \in S})$ ([13]). Actually Q, as a panel space, is modeled on the standard cone in $\mathbb{R}^n$ ([13]). Such a panel space Q is called a *panel manifold*. If furthermore $(Q, (Q_s)_{s \in S})$ is admissible it is called an *admissible S-panel manifold*.

Remark 4.1. The proofs given in [13] are for manifolds without boundary, but the same methods work when W acts by reflections on a manifold with boundary. Let Q be a chamber of such an action. Then Q is an S-panel manifold, and it is also a manifold with boundary

$$\partial Q = \delta Q \cup (Q \cap \partial M).$$

Also, each panel Q_s is an $(S - \{s\})$-panel $(n - 1)$-manifold. Inductively, each Q_J is an $(S - J)$-panel $n - |J|$-manifold with boundary

$$\partial Q_J = \bigcup (\{Q_K : J \subsetneq K, |K| = |J| + 1\}) \cup (Q_J \cap \partial M).$$

Definition 4.2. A *Coxeter W-manifold* is a contractible W-manifold (possibly with boundary) on which W acts properly discontinuously, locally smoothly, cocompactly and so that the fixed point sets of finite subgroups are non-empty and contractible. In particular, a Coxeter W-manifold is a manifold of type $\mathcal{E}W$ ([9]; [27]).

Using the theory of reflection groups on manifolds, M. Davis characterized the Coxeter systems that admit cocompact actions, by reflections, on contractible manifolds. They are the Coxeter systems such that $K_0(W, S)$ is a *generalized homology sphere*, i.e., $K_0(W, S)$ and the link of each simplex are homology spheres. M. Davis' result concerns topological manifolds. There is a discussion for an extension of the results to smooth manifolds in Section 17, [13].

Definition 4.3. A Coxeter system (W, S) is called a *manifold-reflection system* if $K_0(W, S)$ is a generalized homology sphere. The group W is called a *manifold-reflection group*.

Remark 4.2.   1. For a manifold-reflection system (W, S), the universal space $\mathcal{U}(W, |K(W, S)|)$ is a contractible W-manifold ([13]) and a space of type $\mathcal{E}W$ (Proposition 3.6). Thus $\mathcal{U}(W, |K(W, S)|)$ is a Coxeter W-manifold.
   2. In [16], the complete characterization of the virtual Poincaré duality Coxeter groups is given. It is shown that (W, S) is a virtual Poincaré duality group if and only $(W, S) \cong (W_1, S_1) \times (W_2, S_2)$, where W_2 is a finite Coxeter group and (W_1, S_1) is a manifold-reflection system.

In the following lemma we characterize the action of a reflection group on Coxeter manifolds.

Lemma 4.1. *With the above notation:*
   1. *Let M be a Coxeter W-manifold without boundary and (W, S) a manifold-reflection group. Then W acts by reflections on M.*
   2. *Let $(M_i^n, \partial M_i)$, $i = 1, 2$, be Coxeter W-manifolds with boundary. Let W be a manifold-reflection group that acts by reflections on M_2 and ∂M_2. Let*

$$f : (M_1, \partial M_1) \rightarrow (M_2, \partial M_2)$$

> be a W-homotopy equivalence which restricts to a W-homeomorphism on ∂M_1. Then W acts by reflections on M_1.

Furthermore, an element of W acts as a reflection on M_1 if and only if it acts as a reflection on M_2.

Proof. (i) In [26], Proposition 2.14 and Proposition 3.8, the result is proved for W a right angled Coxeter group. The proof extends to the general case. The proof actually shows that a reflection of (W, S) acts as a reflection on M.

(ii) Let S be a set of Coxeter generators induced by the W-action on M_2 (Theorem 2.1). Let $r \in W$ be an element that acts as a reflection. First notice that M_1^r has the same dimension as M_2^r because their boundaries have the same dimension. Thus M_1^r is codimension 1 submanifold of M_1. Let $M'_i = M_i - \partial M_i$, $i = 1, 2$. Then f induces a proper W-homotopy equivalence from M'_1 to M'_2. Thus

$$H_c^n(M'_1, M'^r_1) \cong H_c^n(M'_2, M'^r_2),$$

and Alexander-Poincaré duality implies that

$$H_0(M'_1 - M'^r_1) \cong H_0(M'_2 - M'^r_2) \cong \mathbb{Z} \oplus \mathbb{Z},$$

because s acts as a reflection on M'_2. So M'^r_1 is a codimension one submanifold of M'_1 that separates M'_1 into two components. Thus W is generated by a set of elements that act by reflections on M'_1 and ∂M_1. Therefore W acts by reflections on M_1.

The same argument shows that if $r \in W$ acts as a reflection on M_1, then it acts as a reflection on M_2.

Notice that, in both cases, the W-action on M_1 is effective because the action on M_2 is effective. $\square$

The following lemma holds for any discrete group W.

Lemma 4.2. *Let M be a contractible manifold on which W acts properly discontinuously and cocompactly. Let $W = W_1 \times W_2$ be a decomposition of W such that W_2 is finite and M^{W_2} is contractible. If $\partial M \neq \emptyset$, assume that W_2 acts trivially on ∂M. Then W_2 acts trivially on M.*

Proof. We first assume that $\partial M = \emptyset$. The group $N(W_2)/W_2 \cong W_1$ acts on the fixed point set M^{W_2}. Since M^{W_2}/W_1 is a closed subset of the compact space M/W_1, the action of W_1 on the contractible manifold (without boundary) M^{W_2} is cocompact. Thus

$$\dim(M^{W^2}) = \mathrm{vcd}(W_1) = \mathrm{vcd}(W) = \dim(M).$$

Therefore M^{W_2} is a submanifold without boundary of the connected manifold M of the same dimension, which is closed as a topological subspace. By the invariance of domain, $M^{W_2} = M$, which implies that W_2 acts trivially on M.

If $\partial M \neq \emptyset$, then the fixed point set M^{W_2}, is a submanifold of M with boundary ∂M which is a closed subspace. As before, $M^{W_2} = M$ and W_2 acts trivially. $\square$

Corollary 4.3. *Let (W, S) be a manifold-reflection group. Then there is no decomposition of (W, S) as a product $(W_1, S_1) \times (W_2, S_2)$ with W_2 a finite Coxeter group.*

Proof. Let M be a contractible manifold on which W acts by reflections. Thus W_2 acts by reflections on M, which contradicts Lemma 4.2. $\square$

We will now study uniqueness properties of the Coxeter presentation of a manifold-reflection Coxeter group.

Lemma 4.4. *Let (W, S) be a manifold-reflection system. Then $W_{S-\{s\}}$ has infinite index in W for each $s \in S$. Furthermore, $\mathrm{vcd}(W_{S-\{s\}}) < \mathrm{vcd}(W)$.*

Proof. We will give a proof by contradiction. We assume that $W_{S-\{s\}}$ has finite index in W. The Coxeter system (W, S) can be written as a product of Coxeter systems (W_i, S_i), $i = 1, \ldots k$, such that, for each i, (W_i, S_i) is irreducible. Let $s \in S_{i_0}$. Then there is a bijection between right cosets $W/W_{S-\{s\}}$ and $W_{S_{i_0}}/W_{S_{i_0}-\{s\}}$. Thus $[W_{S_{i_0}} : W_{S_{i_0}-\{s\}}] < \infty$. Proposition 4.2(ix) in [17] implies that $W_{S_{i_0}}$ is finite, contradicting Corollary 4.3.

Since W is a virtual Poincaré duality group, it contains a Poincaré duality subgroup W_0 of finite index. Then the subgroup $W_0 \cap W_{S-\{s\}}$ is a torsion free subgroup of finite index of $W_{S-\{s\}}$, and it is contained in W_0. By [31], $\mathrm{cd}(W_0 \cap W_{S-\{s\}}) < \mathrm{cd}(W_0)$. But

$$\mathrm{vcd}(W_{S-\{s\}}) = \mathrm{cd}(W_0 \cap W_{S-\{s\}}) < \mathrm{cd}(W_0) = \mathrm{vcd}(W).$$

$\square$

Let (W, S) be a Coxeter system which acts on a manifold M^n by reflections. We write Q for the fundamental domain, which is homeomorphic to a manifold with boundary. Write

$$R = \bigcup_{w \in W} wSw^{-1}.$$

Then the elements of R act on M by reflections, and the singular set of the action is given by

$$\sigma(M) = \bigcup_{r \in R} M^r.$$

In what follows, $\mathbb{Z}/2\mathbb{Z}$-coefficients are understood. There is a Poincaré duality isomorphism

$$P : H_c^n(M, \sigma(M)) \to H_0(M - \sigma(M)).$$

But since the action is by reflections,

$$M - \sigma(M) = \bigcup_{w \in W} \mathrm{int}(wQ).$$

By the naturality of the Poincaré duality, the inclusion maps induce an isomorphism

$$\iota_c^* = \bigoplus_{w \in W} \iota_w^* : H_c^n(M, \sigma(M)) \to \bigoplus_{w \in W} H_c^n(wQ, \partial(wQ)).$$

Thus the set $\mathcal{B} = \{(\iota_c^*)^{-1}([wQ, \partial(wQ)]^*)/w \in W\}$ forms a basis of $H_c^n(M, \sigma(M))$ (here $[wQ, \partial(wQ)]^*$ demotes the dual of the Poincaré class of $(wQ, \partial(wQ))$. We write $[wQ]^*$ for $(\iota_c^*)^{-1}([wQ, \partial(wQ)]^*)$. On the other hand, W acts on $H_c^n(M, \sigma(M))$, on the left, in such a way that each element w acts through the map

$$(w^{-1})^* : H_c^n(M, \sigma(M)) \to H_c^n(M, \sigma(M)).$$

Under this action, $H_c^n(M, \sigma(M))$ has the structure of a free left $\mathbb{Z}/2\mathbb{Z}[W]$-module on one generator $[Q]^*$.

Lemma 4.5. *Let $W = \langle r \rangle \cong \mathbb{Z}/2\mathbb{Z}$ act by reflections on manifolds M_i^n, $i = 1, 2$, and let $\mathcal{B}_i$, $i = 1, 2$, be the $\mathbb{Z}/2\mathbb{Z}$-bases of $H_c^n(M_i, M_i^r)$ corresponding to the components of $M_i - M_i^r$. If $f : M_1 \to M_2$ is a proper W-homotopy equivalence, then $f^*(\mathcal{B}_2) = \mathcal{B}_1$.*

Proof. Identify $H_c^n(M_i, M_i^r)$ with the free $\mathbb{Z}/2\mathbb{Z}[W]$-module with basis a fundamental domain $[Q_i]$ $(i = 1, 2)$. Then the map $f^* : H_c^n(M_2, M_2^r) \to H_c^n(M_1, M_1^r)$ is a $\mathbb{Z}/2\mathbb{Z}[W]$-isomorphism. Thus it is given by multiplication by a unit of $\mathbb{Z}/2\mathbb{Z}[W]$. But the units of $\mathbb{Z}/2\mathbb{Z}[W]$ are $\{1, r\}$. Then $f^*([Q_2])$ is equal to $[Q_1]$ or to $r^*([Q_1])$, which implies the result. $\square$

We assume further that the action of W on M is cocompact. In this case, all the domains wQ are compact manifolds with boundary. There is an isomorphism ([19], Ch. VIII, §2)

$$J : H_n(M, \sigma(M)) \to \Gamma_c(M - \sigma(M)),$$

where the term on the right is the vector space generated by the components of $M - \sigma(M)$ that have compact closure. Since all the components have compact closure, the inclusion maps induce an isomorphism

$$\iota_* = \bigoplus_{w \in W} \iota_{w*} : \bigoplus_{w \in W} H_n(wQ, \partial(wQ)) \to H_n(M, \sigma(M)).$$

For each $w \in W$ we have $w_* : H_n(M, \sigma(M)) \to H_n(M, \sigma(M))$, and the collection of the maps w_* induces an action of W on $H_n(M, \sigma(M))$. Under this action, $H_n(M, \sigma(M))$ becomes a free left $\mathbb{Z}/2\mathbb{Z}[W]$-module with generator $[Q]$, the image of the fundamental class of $(Q, \partial(Q))$.

We also consider the dual construction. Taking duals, as $\mathbb{Z}/2\mathbb{Z}$ vector spaces, results to an isomorphism

$$\prod_{w \in W} \iota_w^* : H^n(M, \sigma(M)) \to \prod_{w \in W} H^n(wQ, \partial(wQ)).$$

The induced action of W on $H^n(M, \sigma(M))$ is by the formula $x \cdot w = w^*(x)$ and induces a right $\mathbb{Z}/2\mathbb{Z}[W]$-module structure on it. Since $\mathbb{Z}/2\mathbb{Z}[W]$ is a ring with involution, $H^n(M, \sigma(M)))$ admits a natural left $\mathbb{Z}/2\mathbb{Z}[W]$-module structure with $w \cdot x = x \cdot w^{-1} = (w^{-1})^*(x)$. We would like to characterize the dual of $H_n(M, \sigma(M))$ as a $\mathbb{Z}/2\mathbb{Z}[W]$-module. By [6], Ch. VIII, Lemma 7.4,

$$\mathrm{Hom}_{\mathbb{Z}/2\mathbb{Z}[W]}(H_n(M, \sigma(M)), \mathbb{Z}/2\mathbb{Z}[W]) \cong \mathrm{Hom}_c(H_n(M, \sigma(M)), \mathbb{Z}/2\mathbb{Z}),$$

where the last group consists of $\mathbb{Z}/2\mathbb{Z}$-homomorphisms ϕ from $H_n(M, \sigma(M))$ to $\mathbb{Z}/2\mathbb{Z}$ such that $\phi(wx) = 0$ for all but finitely many w. In this case,

$$\mathrm{Hom}_c(H_n(M, \sigma(M)), \mathbb{Z}/2\mathbb{Z}) \cong \bigoplus_{w \in W} H^n(wQ, \partial(wQ))$$

as left $\mathbb{Z}/2\mathbb{Z}[W]$-modules. But there is a Poincaré duality isomorphism

$$P : H_c^n(M, \sigma(M)) \to H_0(M - \sigma(M)),$$

which implies that

$$\iota_c^* = \bigoplus_{w \in W} \iota_w^* : H_n^c(M, \sigma(M)) \to \bigoplus_{w \in W} H^n(wQ, \partial(wQ))$$

is an isomorphism. Thus the $\mathbb{Z}/2\mathbb{Z}[W]$-dual of $H_n(M, \sigma(M))$ is isomorphic to $H_c^n(M, \sigma(M))$, and the following diagram commutes:

$$
\begin{array}{ccc}
H_c^n(M, \sigma(M)) & \xrightarrow{\oplus_{w \in W} \iota_w^*} & \bigoplus_{w \in W} H^n(wQ, \partial(wQ)) \\
\downarrow & & \downarrow \\
H^n(M, \sigma(M)) & \xrightarrow{\prod_{w \in W} \iota_w^*} & \prod_{w \in W} H^n(wQ, \partial(wQ)).
\end{array}
$$

where the vertical maps are the natural maps.

Combining the above remarks, we get

Lemma 4.6. *The $\mathbb{Z}/2\mathbb{Z}[W]$-vector spaces $H_n(M, \sigma(M))$ and $H_c^n(M, \sigma(M))$ are free $\mathbb{Z}/2\mathbb{Z}[W]$-modules which are naturally dual to each other, i.e., if M and N are two manifolds equipped with a W-action by reflections and $f : M \to N$ is a proper W-map, then $f^* : H_c^n(N, \sigma(N)) \to H_c^n(M, \sigma(M))$ is the $\mathbb{Z}/2\mathbb{Z}[W]$-dual of the map $f_* : H_n(M, \sigma(M)) \to H_n(N, \sigma(N))$.*

Proof. . The first part follows from the remarks before the statement of the lemma. For the naturality property, first notice that the map induced on cohomology is the $\mathbb{Z}/2\mathbb{Z}$-dual of f_*. The result follows by considering the cohomology groups with compact supports as subgroups of the ordinary cohomology and using the commutativity of the last diagram. $\square$

Proposition 4.7. *Let M_i^n, $i = 1, 2$, be Coxeter W-manifolds without boundary. Let W act by reflections on M_2, and let (W, S) be the Coxeter system induced by the action. Then W acts by reflections on M_1, and there is an S-panel submanifold fundamental domain Q_1 of M_1 such that $M_1 \cong_W \mathcal{U}(W, Q_1)$.*

Proof. By Lemma 4.1, W acts by reflections on M_1, and the set of elements of W that act by reflections on M_1 is equal to the set of elements that act by reflections on M_2, which is equal to the set of reflections of (W, S). Choose a fundamental domain P of the W action on M_1 and suppose that T is the set of generating reflections determined by the action t. Then $M_1 \cong_W \mathcal{U}(W, P)$ ([13], Thm. 4.1). From now on, we will assume that M_1 is the universal space $\mathcal{U}(W, P)$. By [25], the manifold M_1 admits a CAT(0) locally flat metric, and W acts on M_1 by isometries.

Since Coxeter manifolds are spaces of type $\mathcal{E}W$, any two are W-homotopy equivalent ([9]). Let $f : M_1 \to M_2$ be a W-homotopy equivalence and g a W-homotopy inverse of f. Since the action is properly discontinuous and cocompact, f is a proper homotopy equivalence. Since W acts by reflections on M_2, the generators $s \in S$ act as reflections on M_2 and there is an S-panel subspace Q_2 of M_2 such that $M_2 \cong_W \mathcal{U}(W, Q_2)$. The subspace Q_2 is the closure of a component of the non-singular set of M_2. Since Q_2 is also the orbit space of the action, it is compact.

For $R' \subseteq R$ a set of reflections in W, set $M_i(R') = \bigcup_{r \in R'} M_i^s$, $i = 1, 2$. In particular, $M_i(R) = \sigma(M_i)$.

Claim 1. $H_n(M_2, M_2(S)) \cong \mathbb{Z}$, and for each $s \in S$, $H_n(M_2, M_2(S - \{s\})) = 0$.

Proof. The nerve of the cover of $M_2(S)$ by the walls M_2^s ($s \in S$) is the same as the nerve of the cover of ∂Q_2 by the panels Q_{2s}. Since all the intersections of the elements of the cover are contractible, the orbit map $p_2 : M_2 \to Q_2$ induces a homotopy equivalence $p_2| : M_2(S) \to \partial Q_2$ ([2], Thm. 8.2.2). Thus

$$H_n(M_2, M_2(S)) \cong H_n(Q_2, \partial Q_2) \cong \mathbb{Z}.$$

For the second part, set $S' = S - \{s\}$, $W' = W_{S'}$. Then W' acts on M_2 by reflections. Let Q_2' be the closure of the component of the singular set of the W' action that contains Q_2. Then, for each $s' \in S'$,

$$\dim(Q_2' \cap M_{2s'}) = \dim(Q_2 \cap M_{2s'}) = n - 1.$$

Thus Q_2' is a fundamental domain of the W' action on M_2. By Lemma 4.4, $\mathrm{vcd}(W') < n$, and thus Q_2' is not compact, which implies

$$H_n(M_2, M_2(S')) \cong H_n(Q_2', \partial Q_2') = 0.$$

Claim 2.   (i) $H_n(M_1, M_1(S)) \cong \mathbb{Z}$, and, for each $s \in S$, $H_n(M_1, M_1(S - \{s\})) = 0$.
(ii) $M_1 - M_1(S)$ contains a unique component with compact closure, but all the components of $M_1 - M_1(S - \{s\})$ have non-compact closure.

Proof. The W-homotopy equivalence f induces a homotopy equivalence of pairs

$$f : (M_1, M_1(S')) \to (M_2, M_2(S')), \qquad S' = S \text{ or } S - \{s\}.$$

The result in (i) follows from Claim 1. For part (ii), notice that there is a sequence of isomorphisms ($S' = S$ or $S' = S - \{s\}$)

$$\Gamma_c(M_2 - M_2(S')) \xrightarrow{J_2^{-1}} H_n(M_2, M_2(S')) \xrightarrow{f_*} H_n(M_1, M_1(S'))$$
$$\xrightarrow{J_1} \Gamma_c(M_1 - M_1(S')),$$

where $\Gamma_c(-)$ denotes the free abelian group on the components of the space which have compact closure ([19], Ch. VIII, Proposition 3.3). The result follows from part (i).

Claim 3. Let Q_1 be the closure of the unique component of $M_1 - M_1(S)$ with compact closure. Then:

(i) Q_1 is homeomorphic to a compact manifold with boundary, and it is a totally geodesic subspace of M_1.
(ii) $\dim(Q_1 \cap M_1^s) = n - 1$, for all $s \in S$.
(iii) The space Q_1 is a union of an odd number of domains, i.e.

$$Q_1 = \bigcup_{j=1}^{2k+1} w_j P.$$

(iv) Set

$$[Q_1]^* = \sum_{j=1}^{2k+1} [w_j P]^* \in H_c^n(M_1, \sigma(M_1))$$

($\mathbb{Z}/2\mathbb{Z}$-coefficients are understood). Then $f^*([Q_2]^*) = [Q_1]^*$.

Proof. (i) For each $s \in S$, the fixed point set M_1^s is a totally geodesic subspace of M_1. Furthermore, $M_1 - M_1^s = \mathrm{int}(M_1^s(+)) \cup \mathrm{int}(M_1^s(-))$ as a union of two components, and the closure of each component is a totally geodesic subspace of M_1. The space Q_1 is the intersection of the half spaces that contain it, i.e. it is the intersection of totally geodesic subspaces and thus it is totally geodesic. Since $s \in S$ acts by reflections on M_1, Q_1 is a manifold with boundary.

We prove (ii) by contradiction. Let $s \in S$ be such that $\dim(Q_1 \cap M_1^s) < n - 1$. Then the set $M_1 - M_1(S - \{s\})$ has a component with compact closure Q_1, contradicting Claim 2(ii).

For (iii), we use homology and cohomology with $\mathbb{Z}/2\mathbb{Z}$-coefficients. The group $H_n(M_i, M_i(S))$ is generated by the fundamental class of $(Q_i, \partial Q_i)$, for $i = 1, 2$. Since

$$f_* : H_n(M_1, \sigma(M_1)) \to H_n(M_2, \sigma(M_2))$$

is an isomorphism of free $\mathbb{Z}/2\mathbb{Z}[W]$-modules in one generator (Lemma 4.6), there is a unit u in $\mathbb{Z}/2\mathbb{Z}[W]$ such that $f_*([P]) = u([Q_2])$. The element u is the sum of an odd number of elements of the group ring, because u is mapped to the unit of $\mathbb{Z}/2\mathbb{Z}$ under the augmentation map. The same is true for the inverse u^{-1}. Now let

$$f_{S*} : H_n(M_1, M_1(S)) \to H_n(M_2, M_2(S)),$$

and each group is a $\mathbb{Z}/2\mathbb{Z}$-vector space of dimension 1. Thus $f_{S*}([Q_1, \partial Q_1]) = [Q_2, \partial Q_2]$. By the commutativity of the diagram

$$
\begin{array}{ccc}
H_n(M_1, M_1(S)) & \xrightarrow{f_{S*}} & H_n(M_2, M_2(S)) \\
\kappa_1 \downarrow & & \downarrow \kappa_2 \\
H_n(M_1, \sigma(M_1)) & \xrightarrow{f_*} & H_n(M_2, \sigma(M_2))
\end{array}
$$

we derive that

$$
\begin{aligned}
[Q_2] = \kappa_2 \circ f_{S*}([Q_1]) &= f_* \circ \kappa_1([Q_1]) \\
&\Rightarrow \kappa_1([Q_1]) = (f_*)^{-1}([Q_2]) = u^{-1}[P] \\
&\Rightarrow \kappa_1([Q_1]) = \sum_{j=1}^{2k+1} [w_j P].
\end{aligned}
$$

By the definition of the map κ_1,

$$
Q_1 = \bigcup_{j=1}^{2k+1} w_j P.
$$

Part (iv) follows from the fact that f^* is the $\mathbb{Z}/2\mathbb{Z}[W]$-dual of the map f_* (Lemma 4.6).

Claim 4. For each $R' \subset R$, the *natural set of generators* of $H_c^n(M_i, M_i(R'))$ is the set corresponding to the components of $M_i - M_i(R')$ under the Poincaré duality isomorphism ($i = 1, 2$). Then for each $r \in R$, the inclusion induced map

$$
\iota_{ir} : H_c^n(M_i, \sigma(M_i)) \rightarrow H_c^n(M_i, M_i^r), \quad i = 1, 2,
$$

maps $[wQ_1]^*$ to a natural generator. Furthermore, the diagram commutes:

$$
\begin{array}{ccc}
H_c^n(M_1, \sigma(M_1)) & \xrightarrow{f^*} & H_c^n(M_2, \sigma(M_2)) \\
\iota_{1r} \downarrow & & \downarrow \iota_{2r} \\
H_c^n(M_1, M_1^r) & \xrightarrow{f_r^*} & H_c^n(M_2, M_2^r)
\end{array}
$$

In particular, if $M_i^r(+)$ demotes the closure of the component of $M_i - M_i^r$ such that $\iota_{ir}([Q_i]^*) = [M_i^r(+)]^*$ ($i = 1, 2$, $r \in R$), then $f_r^*([M_1^r(+)]^*) = [M_2^r(+)]^*$.

Proof. By Claim 3, we know that

$$
wQ_1 = \bigcup_{j=1}^{2k+1} w w_j P.
$$

Let $M_1 - M_1^r = M_1(+) \cup M_1^r(-)$, the union of two closed half-spaces whose intersection in M^r. We consider two cases

Case 1. Let wQ_1 be contained in the closure of one of the half-spaces determined by r, say $M_1^r(+)$. Then $w w_j P$ is contained in $M_1^r(+)$ for $j = 1, \ldots, 2k+1$. If $[M_1^r(+)]^*$ denotes the natural generator of $H_c^n(M_1, M_1^r)$ represented by $M_1^r(+)$, then

$$
\iota_{1r}([w w_j P]^*) = [M_1^r(+)]^* \Rightarrow \iota_{1r}([wQ_1]^*) = \sum_{j=1}^{2k+1} \iota_{1r}([w w_j P]^*) = [M_1^r(+)]^*.
$$

Case 2. If wQ_1 is not contained in any of $M_1^r(\pm)$, then $M_1^r \cap \text{int}(wQ_1) \neq \emptyset$. Since wQ_1, $M_1^r(\pm)$ are totally geodesic subspaces of M_1, the intersections $wQ_1 \cap M_1^r(\pm)$ are totally geodesic and therefore connected. Each intersection is a union of chambers. Since the number of chambers in wQ_1 is odd, one intersection will contain

an odd number of chambers and the other an even number. Then, $\iota_{1r}([wQ_1]^*)$ is equal to the generator corresponding to the half-space containing an odd number of chambers.

The last assertion of the claim follows from Lemma 4.5.

Claim 5. For each $r \in R$, define
$$P_r' = \{w \in W : \iota_{1r}([wQ_1]^*) = \iota_{1r}([Q_1]^*)\}.$$
Then $P_r' = P_r = \{w \in W : \ell(rw) > \ell(w)\}$, where the length is relative to the generating set S.

Proof. Claim 4 implies that the element $\iota_{1r}([wQ_1]^*)$ is a natural generator, for all $w \in W$. Since the action of W on M_2 is by reflections with fundamental domain Q_2, P_r is the set of all $w \in W$ such that wQ_2 and Q_2 belong to the same half-space bounded by M^r ([13], Remark 7.4) or equivalently, $\iota_{2r}([wQ_2]^*) = \iota_{2r}([Q_2]^*)$. Since the diagram

$$
\begin{array}{ccc}
H_c^n(M_1, \sigma(M_1)) & \xrightarrow{\;g^*\;} & H_c^n(M_2, \sigma(M_2)) \\
\Big\downarrow{\scriptstyle \iota_{1r}} & & \Big\downarrow{\scriptstyle \iota_{2r}} \\
H_c^n(M_1, M_1^r) & \xrightarrow{\;g^*\;} & H_c^n(M_2, M_2^r)
\end{array}
$$

commutes, we have the following:

$$
\begin{aligned}
w \in P_r' &\Leftrightarrow \iota_{1r}([wQ_1]^*) = \iota_{1r}([Q_1])^* \\
&\Leftrightarrow g^* \iota_{1r}([wQ_1]^*) = g^* \iota_{1r}([Q_1]^*) \\
&\Leftrightarrow \iota_{2r} g^*([wQ_1]^*) = \iota_{2r} g^*([Q_1]^*) \\
&\Leftrightarrow \iota_{2r}([wQ_2]^*) = \iota_{2r}([Q_2]^*)
\end{aligned}
$$

(the last implication is a consequence of Claim 3(iv) and the fact that g^* is W-equivariant), which means that wQ_2 and Q_2 are on the same side of M_2^r.

Claim 6. Let $s, s' \in S$, $s \neq s'$. Then $M_1^s(+) \cap M_1^{s'}(+)$ is a fundamental domain of the action of the dihedral group $\langle s, s' \rangle$.

Proof. Let r be a reflection in $\langle s, s' \rangle$ of shortest length (in the generators s and s') such that $M_1^r \cap (M_1^s(+) \cap M_1^{s'}(+)) \neq \emptyset$. We can assume that the order of ss' is greater than two, because if it is equal to 2, there are no reflections in $\langle s, s' \rangle$ other than s and s'. Without loss of generality we can assume that $r = s'rs'$ is a reduced expression of r. We will give the proof in the case that the dihedral group generated by s and s' is finite. The other case follows similarly. The space M_1^r separates $M_1^s \cap (M_1^s(+) \cap M_1^{s'}(+))$ and $M_1^{s'} \cap (M_1^s(+) \cap M_1^{s'}(+))$. Since Q_1 intersects both intersections in a codimension 1 submanifold, $M_1^r \cap \mathrm{int}(Q_1) \neq \emptyset$. Let
$$\iota_i : H_c^n(M_i, \sigma(M_i)) \to H_c^n(M_i, M_i(\{s, s', r\})), \quad i = 1, 2,$$
be the inclusion induced maps. The following diagram commutes:

$$
\begin{array}{ccc}
H_c^n(M_2, \sigma(M_2)) & \xrightarrow{\;f^*\;} & H_c^n(M_1, \sigma(M_1)) \\
\Big\downarrow{\scriptstyle \iota_2} & & \Big\downarrow{\scriptstyle \iota_1} \\
H_c^n(M_2, M_2(\{s, s', r\})) & \xrightarrow{\;f'\;} & H_c^n(M_1, M_1(\{s, s', r\}))
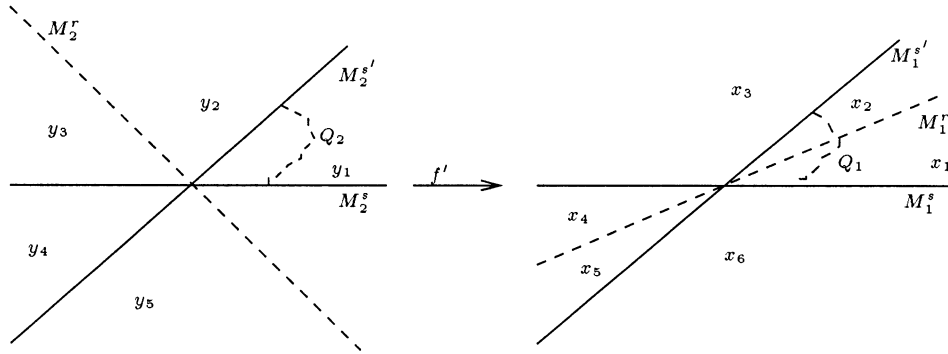\end{array}
$$

FIGURE 1. Schematic representation of the regions deteremined by the three walls M_i^s, $M_i^{s'}$, M_i^r.

(where f' is induced by f) and the two horizontal maps are isomorphisms. We represent the generators of the groups $H_c^n(M_i, M_i(\{s, s', r\}))$, $i = 1, 2$, schematically in Figure 1.

If $s'(\text{int}(Q_1)) \cap M_1^r \neq \emptyset$, then $\text{int}(Q_1) \cap M_1^{r'} \neq \emptyset$, contradicting the minimality of the length of r. Thus $s'(\text{int}(Q_1)) \cap M_1^r = \emptyset$ and $s'Q_1 \subset x_3 \cup x_4$. Since $s'Q_1$ is the union of an odd number of domains and M_1^s separates it into two components, either x_3 or x_4 contains an odd number of components. Then $\iota_1([s'Q_1]^*)$ is equal to either x_3 or x_4. If $\iota_1([s'Q_1]^*) = x_4$, then $\iota_{1s}([s'Q_1]^*) = [M_1^s(-)]^*$, contradicting Claim 5. Thus $\iota_1([s'Q_1]^*) = x_3$, which implies that $f'(y_2) = x_3$. Also, $rs'Q_1 \subset x_1 \cup x_5 \cup x_6$, which implies that $\iota_1([rs'Q_1]^*) = a_1 x_1 + a_5 x_5 + a_6 x_6$.

If $\ell(srs') < \ell(rs')$, then $rs' = sr''$, which cannot happen in a dihedral group for a reflection r. Thus $\ell(srs') > \ell(rs')$. In particular, $\iota_2([rs'Q_2]^*) = y_3$ (because $rs'Q_2$ must lie on the opposite side of $M_2^{s'}$ and M_2^r, and on the same side of M_2^s as Q_2). Then

$$\begin{aligned}
f'(y_3) &= f'(\iota_2([rs'Q_2]^*)) \\
&= \iota_1(f^*([rs'Q_2]^*)) \\
&= \iota_1([rs'Q_1]^*) \\
&= a_1 x_1 + a_5 x_5 + a_6 x_6
\end{aligned}$$

Let

$$j_i : H_c^n(M_i, M_i(\{s, s', r\})) \rightarrow H_c^n(M_i, M_i^s \cup M_i^{s'})$$

be the inclusion induced map ($i = 1, 2$). Then the following diagram commutes:

$$\begin{array}{ccc}
H_c^n(M_2, M_2(\{s, s', r\})) & \xrightarrow{f'} & H_c^n(M_1, M_1(\{s, s', r\})) \\
\downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle j_1} \\
H_c^n(M_2, M_2(\{s, s'\})) & \xrightarrow{f''} & H_c^n(M_1, M_1(\{s, s'\}))
\end{array}$$

Using the fact that $j_2(y_2) = j_2(y_3)$, we have

$$f''(j_2(y_2)) = f''(j_2(y_3))$$
$$\Rightarrow j_1(f'(y_2)) = j_1(f'(y_3))$$
$$\Rightarrow j_1(x_3) = j_1(a_1 x_1 + a_5 x_5 + a_6 x_6)$$

which cannot occur. Contradiction.

For the case of the infinite dihedral group, the same method works. In this case, the proof actually is outlined in [33], Proposition 4, part (ii).

Claim 7. The space Q_1 is a fundamental domain of the W-action on M_1.

Proof. The proof of the Claim 7 is classical. Claim 6 implies that $M_1^s(+) \cap M_1^{s'}(+)$ is the fundamental domain of the action of the dihedral group generated by s and s'. Thus, for all w in the dihedral group,

(i) $w(M_1^s(+) \cap M_1^{s'}(+)) \subset M_1^s(+)$ and $\ell(sw) = \ell(w) + 1$, or
(ii) $w(M_1^s(+) \cap M_1^{s'}(+)) \subset sM_1^s(+)$ and $\ell(sw) = \ell(w) - 1$.

The rest of the proof follows as in [23], Theorem 2.1 (also [32], [33]).

Thus Q_1 is the closure of a component of the non-singular set. Since $\dim(Q_1 \cap M_1^s) = n - 1$ (for all $s \in S$), it contains points which are fixed only by s. By definition Q_1 is an S-panel fundamental domain for the W-action on M_1. □

Combining Proposition 3.6 and Proposition 4.7, we have a uniqueness statement on the manifold-reflection presentation of a Coxeter group.

Theorem 4.8. *Let (W, T) and (W, S) be two manifold-reflection systems in W. Then there is an inner automorphism of W that maps T to S, i.e., T and S are conjugate.*

Proof. By Theorem 15.3 in [13], there are two contractible manifolds M_T and M_S such that W acts cocompactly by reflections on both of them, and the boundary of the fundamental domain is $|K_0(W, T)|$ in the first case and $|K_0(W, S)|$ in the second (as panel manifolds). The manifolds M_T and M_S are both spaces of type $\mathcal{E}W$ by Proposition 3.6. Thus they are Coxeter manifolds. Write C for the chamber in M_T with boundary panel homeomorphic to $|K_0(W, T)|$. By Proposition 4.7, M_T admits a fundamental domain C', with boundary panel homeomorphic to $|K_0(W, S)|$. Thus M_T admits two chambers, one determining the Coxeter generators T and the other the Coxeter generators S. By Theorem 4.1(iii) in [13], there is an element $w \in W$ that maps one chamber to the other. Then conjugation by w induces a bijection between S and T. □

Remark 4.3. In a recent preprint ([8]), R. Churney and M. Davis have proved the result in Theorem 4.8 using more combinatorial methods. Actually their methods produce a more general result. More specifically, they derive the same uniqueness result by assuming that the Coxeter group admits an action by reflections on certain pseudo-manifolds that are not necessarily manifolds.

5. Structure Set Rigidity Theorems

In this section we are going to prove the rigidity theorems for the structure set of manifolds equipped with a Coxeter group action, not necessarily by reflections.

Let M be a Coxeter manifold without boundary. Since W acts properly discontinuously and cocompactly on M, W is a virtual Poincaré duality group. By Theorem B in [16], $(W, S) \cong (W_1, S_1) \times (W_2, S_2)$, where W_2 is a finite Coxeter group and (W_1, S_1) is a manifold-reflection system. Notice that M is a Coxeter W_1-manifold. By Lemma 4.2, the group W_2 acts trivially on M.

We need the following lemma from surgery theory, which is derived using the Poincaré conjecture ([26], Lemma 3.1; [35]; [20], for $n = 4$).

Lemma 5.1. *Let $(M^n, \partial M)$ be a compact contractible manifold with boundary. Then the relative structure set*

$$\mathcal{S}(M^n, \partial M) = *   \text{ for } n \neq 3.$$

The next lemma will be used in the construction of homeomorphisms of faces of dimension 3.

Lemma 5.2. *Let $(M, \partial M)$ be a compact three dimensional contractible manifold embedded into S^3. Then $(M, \partial M)$ is homeomorphic to the pair (D^3, S^2).*

Proof. The boundary of M is a surface of genus zero. Therefore ∂M is homeomorphic to S^2. By the Schoenflies lemma ([7]), $S^3 - S^2$ consists of two components whose closures are homeomorphic to the 3-disc. Therefore M is homeomorphic to a 3-disc. □

We can now prove the structure set rigidity theorem for Coxeter groups. As always, W will be a Coxeter group. The setting will be the following:

Let M^n be a Coxeter W-manifold without boundary. Let N^{n+k} be a Coxeter W-manifold, and let

$$f : (N, \partial N) \rightarrow (M \times D^k, M \times S^{k-1})$$

be a W-map such that $f|\partial N$ is a W-homeomorphism (the W-action on D^k is trivial).

We need the following assumption on the three dimensional fixed point sets

Definition 5.1. Let Γ be a discrete group acting on a manifold X. The pair (Γ, X) satisfies *the dimension three condition* if the three dimensional fixed point sets of X can be embedded into S^3.

Theorem 5.3. *Let (W, S) be a Coxeter system and f a W-map as above. Assume that (W, N) and $(W, M \times D^k)$ satisfy the dimension three condition. Then f is W-homotopic to a W-homeomorphism.*

Proof. By assumption both spaces N and $M \times D^k$ are spaces of type $\mathcal{E}W$. Thus f is a W-homotopy equivalence. Since M is a Coxeter manifold, W is a virtual Poincaré duality group. Thus there is a decomposition $(W, S) \cong (W_1, S_1) \times (W_2, S_2)$ with (W_1, S_1) a manifold-reflection system and W_2 a finite Coxeter group. By Lemma 4.2, W_2 acts trivially on M. By Lemma 4.1(1), W_1 acts by reflections on M, and by Theorem 4.8, there is an S-panel manifold which is the fundamental domain of the W_1-action on M, which is admissible (Proposition 3.6). Since the action of W_1 on D^k is trivial, W_1 acts by reflections on $M \times D^k$ as well as on $M \times S^{k-1}$. Then $Q_1 = Q \times D^k$ is a fundamental domain of the W_1-action on $M \times D^k$, which is also admissible.

By Lemma 4.2, W_2 acts trivially on N. Therefore it is enough to show that f is W_1-homotopic to a homeomorphism. For the rest of the proof we assume that (W, S) is a manifold-reflection system.

By Lemma 4.1(2), W acts by reflections on N. Let Q_2 be the T-panel fundamental domain determined by the action and $N \cong_W \mathcal{U}(W, Q_2)$.

Claim 1. (W, T) is a manifold-reflection system.

Proof. By Theorem B in [16], (W, T) splits as the product of (W_1, T_1) and (W_2, T_2), where (W_1, T_1) is a manifold-reflection system and (W_2, T_2) is finite. By Lemma 4.2, W_2 acts trivially on N. Unless W_2 is the trivial group, this is impossible because W acts on N by reflections. Thus (W, T) is a manifold-reflection system.

By Theorem 4.8, T and S are conjugate in W. Thus, by choosing another chamber in N, we can assume that Q_2 is an S-panel space and it is admissible by Proposition 3.6.

Since there are W-homeomorphisms

$$M \times D^k \cong_W \mathcal{U}(W, Q_1), \qquad N \cong_W \mathcal{U}(W, Q_2),$$

f induces a W-homotopy equivalence:

$$g : \mathcal{U}(W, Q_2) \to \mathcal{U}(W, Q_1).$$

We will construct an S-panel homeomorphism $\lambda : Q_2 \to Q_1$. The construction is done inductively on the faces. We first show how to extend the function to three dimensional panels.

Claim 2. Let Q_{iL} $(i = 1, 2)$ be 3-dimensional faces and

$$\lambda : (Q_{2L}, \partial Q_{2L}) \to (Q_{1L}, \partial Q_{1L})$$

an S-panel homotopy equivalence such that $\lambda | \partial Q_{2L}$ is a homeomorphism. Then λ is homotopic to a homeomorphism relative to ∂Q_{2L}.

Proof. By the definition of faces, Q_{iL} is contained in a fixed-point set of dimension 3 $(i = 1, 2)$. Since the pairs $(W, M \times D^k)$ and (W, N) satisfy the dimension three condition, Q_{iL} is a compact contractible 3-manifold in S^3. Lemma 5.2 implies that $(Q_{iL}, \partial Q_{2L})$ is homeomorphic to (D^3, S^2). Since every homeomorphism of S^2 extends to a homeomorphism on D^3, $\lambda | \partial Q_{2L}$ extends to a homeomorphism λ on Q_{2L} which is homotopic to λ.

Now we start the induction argument. Let Q_{iK}, $i = 1, 2$, be faces of lowest dimension. Then there is a finite subgroup H of W such that

$$Q_{1K} = M^H \times D^k, \quad Q_{2K} = N^H.$$

The subgroup H is a maximal isotropy subgroup of the W-action. If $k = 0$, then Q_{iK} $(i = 1, 2)$ is a contractible manifold without boundary; thus Q_{iK} is a point. Therefore, Q_{1K} and Q_{2K} are homeomorphic. If $k > 0$, then Q_{iK} $(i = 1, 2)$ is a contractible manifold with boundary. The boundaries are

$$\partial Q_{1K} = M^H \times S^{k-1}, \quad \partial Q_{2K} = (\partial N)^H.$$

But $f | \partial Q_{2K}$ is a homeomorphism by assumption. Since the pair $(Q_{2K}, \partial Q_{2K})$ satisfies the homotopy extension property and Q_{1K} is contractible, we can extend $f | \partial Q_{2K}$ to a homotopy equivalence λ_K on Q_{2K}. By Lemma 5.1, λ_K is homotopic (relative to the boundary) to a homeomorphism (we use Claim 2 if the dimension is equal to three).

In general, the boundary of a face Q_{2J} is $\delta Q_{2J} \cup (Q_{2J} \cap \partial N)$. By induction the homeomorphism $\lambda_{\partial Q_{2J}}$ has been defined on δQ_{2J} (which is a union of panels of smaller dimension) and we can set λ to be f when restricted to $Q_{2J} \cap \partial N$. We extend $\lambda_{\partial Q_{2J}}$ to a homotopy equivalence $\lambda'_{Q_{2J}}$ on Q_{2J}. By Lemma 5.1 (or by Claim 2 if the dimension of Q_{2J} is 3) $\lambda'_{Q_{2J}}$ is homotopic to a homeomorphism $\lambda_{Q_{2J}}$. By repeating the construction, we get an S-panel homeomorphism $\lambda : Q_2 \to Q_1$. Then $\mathcal{U}(W, \lambda) \simeq_W g$ ([9]) and $\mathcal{U}(W, \lambda)$ is a W-homeomorphism (Section 2). Therefore

$$f \simeq_W g \simeq_W \mathcal{U}(W, \lambda)$$

The last map is a W-homeomorphism.   □

6. K-theory Rigidity Theorems

We will extend the constructions developed in Section 5 to show rigidity of equivariant h-cobordisms induced by Coxeter groups acting on contractible manifolds. More precisely, let W be a Coxeter group and M a Coxeter W-manifold without boundary. Let W_0 be a normal torsion free subgroup of finite index, and let $G = W/W_0$. Then the manifold $M_0 = M/W_0$ is a closed G-manifold. We will show that the Whitehead group that classifies isovariant G-h-cobordisms over $M_0 \times T^n$ vanishes, i.e., $Wh_G^{Top, \rho}(M_0 \times T^n) = 0$ ([28], [29]).

Let M be a Coxeter W-manifold without boundary, and T^n the n-dimensional torus with the trivial W-action. Then $(W, S) = (W_1, S_1) \times (W_2, S_2)$, where (W_1, S_1) is a manifold-reflection system and W_2 is a finite Coxeter group. The subgroup W_1 acts on M by reflections (Lemma 4.1). The fundamental domain Q of the action is an admissible S-panel manifold. Thus W_1 acts by reflections on $M \times T^n$ with fundamental domain $Q \times T^n$.

Definition 6.1. An S-panel h-cobordism $(X; Q_1, Q_2)$ consists of a W-finite S-panel manifold $(X, (X_s)_{s \in S})$ such that $(X_J; Q_{1J}, Q_{2J})$ is an h-cobordism, for each $J \in \mathcal{F}(W, S)$. An S-h-cobordism X is called *trivial* if X is S-panel homeomorphic to $Q_1 \times I$ with the natural product panel structure.

Lemma 6.1. *Let $\tau = (X; Q_1 \times T^n, Q_2)$ be an S-panel cobordism which is trivial when restricted to faces of dimension less than 4. If Q_1 is an admissible panel manifold, then τ is trivial.*

Proof. Inductively, we extend the trivialization over each face, as in the proof of Theorem 5.3 (in dimension 4 we use Theorem 7.1A in [21]). The first step of the induction works because $Wh(\pi_1(T^n)) = 0$.   □

The next lemma is the analogue of Theorem 4.8 for W-isovariant maps. The proof is easier in this case.

Lemma 6.2. *Let $M_i^{n_i}$, $i = 1, 2$, be two W-manifolds on which W acts properly discontinuously and cocompactly. Assume that W acts on M_2 by reflections and (W, S) is a Coxeter system induced on W by the action. We also assume that M_1^s is a codimension 1 submanifold of M_1, for each $s \in S$. Let $f : M_1 \to M_2$ be an isovariant homotopy equivalence. Then there is an S-panel manifold Q_1 such that $M_1 \cong_W \mathcal{U}(W, Q_1)$.*

Proof. The isovariance of f implies that W acts by reflections on M_1. Let g be an isovariant homotopy inverse of f. Since g is isovariant, it induces a homotopy

equivalence:

$$g| : M_2 - \bigcup_{r \in R} M_2^r \to M_1 - \bigcup_{r \in R} M_1^r,$$

where R is the set of reflections in (W, S). Let Q_1 be the closure of the component corresponding to Q_2. For $s \in S$, Q_2 contains elements fixed only by s. Since g is isovariant, Q_1 contains elements fixed only by s. Thus Q_1 is an S-panel fundamental domain of M_1. $\qquad\square$

Let $h = (N_0; M_0 \times T^n, M_0')$ be an isovariant G-h-cobordism representing an element in $Wh_G^{Top,\rho}(M_0 \times T^n)$ ([28]). Then h is an isovariant G-h-cobordism. We assume that h is a trivial h-cobordism on fixed point sets of dimension less than 4. The universal cover of h is a W-h-cobordism $\tilde{h} = (N; M \times T^n, M')$. By Lemma 4.2, the finite subgroup W_2 of W acts trivially on M and thus on all the spaces involved. So we can assume as before that (W, S) is a manifold-reflection system. Since M' is isovariantly W-homotopy equivalent to $M \times T^n$, and W acts by reflections on $M \times T^n$, W acts by reflections on M'. Let Q' be the S-panel manifold which is the fundamental domain of the action (Lemma 6.2).

Lemma 6.3. *The W-cobordism $\tilde{h}$ is W-trivial.*

Proof. Let $s \in S$. The triple $(N^s, M^s \times T^n, M'^s)$ is an h-cobordism, and thus N^s is a codimension 1 submanifold of N, W-isovariantly homotopic to $M \times T^n$. By Lemma 6.2, W acts by reflections on N and there is an S-panel fundamental domain X of the action. Then $\tau = (X; Q \times T^n, Q')$ is an S-panel h-cobordism which satisfies the assumptions of Lemma 6.1. Thus τ is trivial, i.e. X is S-panel homeomorphic to $Q \times T^n$. Therefore

$$N \cong_W \mathcal{U}(W, X) \cong_W \mathcal{U}(W, Q \times T^n \times I) \cong_W \mathcal{U}(W, Q) \times T^n \times I \cong_W M \times T^n \times I.$$

and the cobordism $\tilde{h}$ is W-trivial. $\qquad\square$

Theorem 6.4. *If M_0 and G are as in the beginning of the section, then*

$$Wh_G^{Top,\rho}(M_0 \times T^n) = 0.$$

Proof. Let $h = (N_0; M_0, M_0')$ be a G-h-cobordism representing an element in the Whitehead group. By Lemma 6.3, the universal cover of h, $\tilde{h} = (N; M \times T^n, M')$, is W-trivial, i.e. $N \cong_W M \times T^n \times I$. Taking the orbit spaces under the W_0 action, we conclude that h is trivial. $\qquad\square$

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